

THE DEHN FUNCTION OF RICHARD THOMPSON'S GROUP T

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ABSTRACT. We improve Guba's result about the Dehn function of R.Thompson's group T and get that $\Phi_T(n) \preceq n^5$, where Φ_T is the Dehn function of group T .

1. INTRODUCTION

By F_s we denote the free group on $a_1, a_2, a_3, \dots, a_s$. Let N be the normal closure of the defining relators R_1, R_2, \dots, R_t in the free group F_s and H be the quotient group F_s/N . For any word $w \in N$, its area is defined by the minimal number

$$\text{Area}(w) = \min\{k | w = \prod_{i=1}^k U_i^{-1} R_{j_i} U_i\},$$

where $U_i \in F_s$, $R_{j_i} \in \{R_1, R_2, \dots, R_t\}^{\pm 1}$. Using van Kampen's Lemma [2, 11], the area of w is equal to the smallest number of cells in a van Kampen diagram if its boundary label is w . If two words v, w are equal in H then we denote by $\|v = w\|$ the number $\text{Area}(vw^{-1})$. The Dehn function Φ is the map from \mathbb{N} to itself defined as

$$\Phi(n) = \max\{\text{Area}(w) | w \in N, |w| \leq n\},$$

where $|w|$ is the length of the word w .

The R.Thompson's group F was discovered by Richard Thompson in the 1960s. The group F is torsion-free, and it can be defined by the following presentation

$$\langle x_0, x_1, x_2, \dots | x_j^{x_i} = x_{j+1} (i < j) \rangle,$$

where $x_j^{x_i} = x_i^{-1} x_j x_i$. F is finitely presented, it can be given by

$$\langle x_0, x_1, | x_2^{x_1} = x_3, x_3^{x_1} = x_4 \rangle,$$

where $x_n = x_0^{-(n-1)} x_1 x_0^{n-1}$ for any $n \geq 2$. For Richard Thompson's group T , it has the following finite presentation:

$$\langle x_0, x_1, c_1, | x_2^{x_1} = x_3, x_3^{x_1} = x_4, c_1 = x_1 c_2, c_2 x_2 = x_1 c_3, c_1 x_0 = c_2^2, c_1^3 = 1 \rangle,$$

where $x_n = x_0^{-(n-1)} x_1 x_0^{n-1}$, $c_n = x_0^{-(n-1)} c_1 x_1^{n-1}$ for any $n \geq 2$.

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Let f, g be two non-decreasing functions from \mathbb{N} to itself. $f \preceq g$ means that there exists a positive integer constant C such that $f(n) \leq Cg(Cn) + Cn$ for all n . $f \sim g$ if and only if $f \preceq g$ and $g \preceq f$.

For a finitely presented group there exists a unique Dehn function, modulo the equivalence relation \sim . The group has solvable word problem if and only if its Dehn function has a recursive upper bound [9, 3, 1]. A finitely presented group is word-hyperbolic group if and only if its Dehn function $\sim n$ [10].

In 1997, Guba and Sapir [4] proved that the Dehn function $\Phi_F(n)$ of group F is strictly subexponential, namely $\Phi_F(n) \preceq n^{\log n} = 2^{\log^2 n}$. After one year, Guba [5] proved that the Dehn function of F satisfies $\Phi_F(n) \preceq n^5$. Finally, Guba [7] proved that $\Phi_F(n) \sim n^2$ in 2006.

For R.Thompson's group T , Guba [6] proved that $\Phi_T(n) \preceq n^7$. We improve this result as the following:

Theorem 1.1. The Dehn function of R.Thompson's group T satisfies polynomial upper bound and $\Phi_T(n) \preceq n^5$.

2. PRELIMINARIES

In this section, we improve some inequalities by using van Kampen diagrams which shall be used to estimate the Dehn function of T in next section.

Lemma 2.1. For any $0 < k \leq n$,

(1)

$$\|c_n = x_n c_{n+1}\| = 1; \quad (2.2)$$

(2)

$$\|c_n x_k = x_{k-1} c_{n+1}\| = O(n^2); \quad (2.3)$$

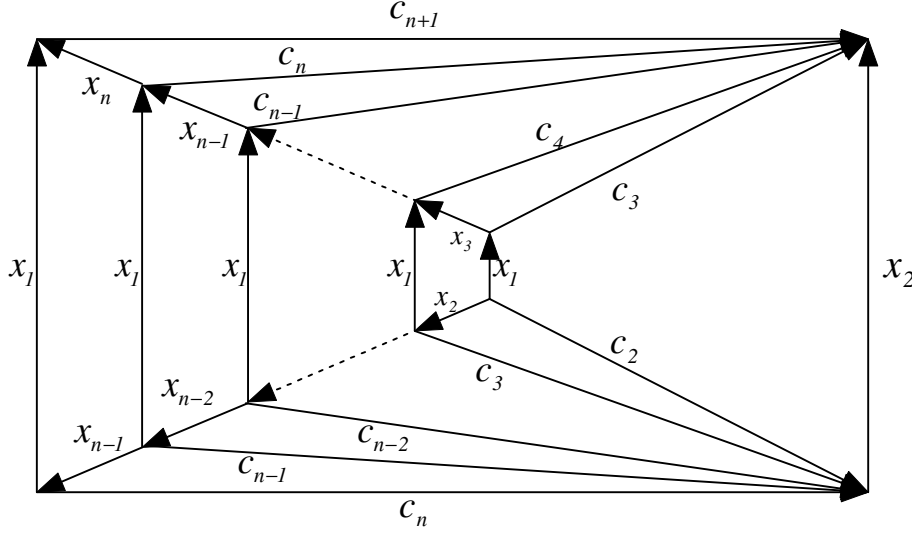
(3)

$$\|c_n x_0 = c_{n+1}^2\| = O(n^3). \quad (2.4)$$

Proof. For (2.2), we just use the relation $c_1 = x_1 c_2$ one time,

$$c_n = x_0^{-(n-1)} c_1 x_1^{n-1} = x_0^{-(n-1)} x_1 c_2 x_1^{n-1} = x_0^{-(n-1)} x_1 x_0^{n-1} x_0^{-(n-1)} c_2 x_1^{n-1} = x_n c_{n+1}.$$

Now let us prove (2.3). Let $h(n, k) = \|c_n x_k = x_{k-1} c_{n+1}\|$. It is easy to see that $h(n, 1) = 0$, $h(2, 2) = 1$. At first, we prove the case $n \geq 3$, $k = 2$. Consider the following van Kampen diagram:



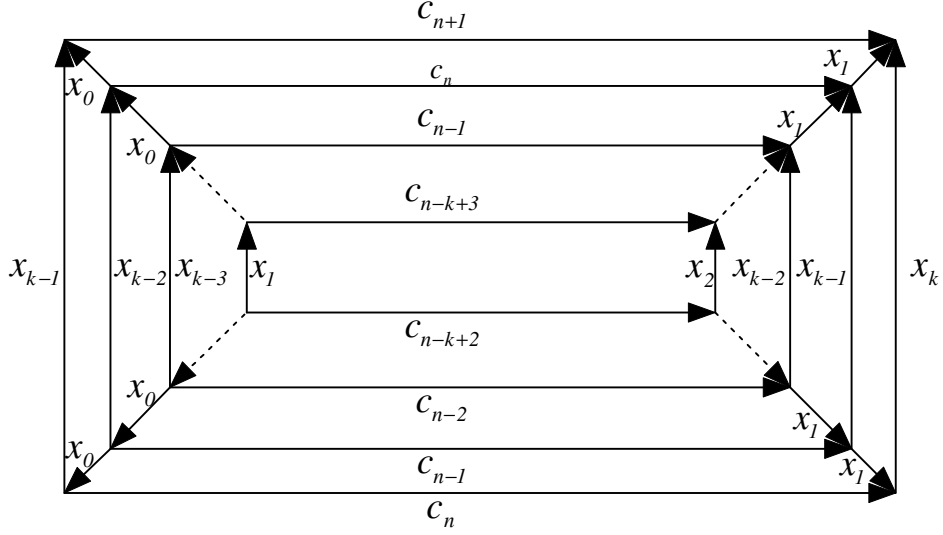
applying (2.2) the above diagram shows that

$$\begin{aligned} h(n, 2) &\leq h(2, 2) + n - 2 + n - 2 + \|x_2 x_3 \cdots x_{n-2} x_{n-1} x_1 = x_1 x_3 \cdots x_{n-1} x_n\| \\ &= h(2, 2) + 2n - 4 + \|x_0^{-1} x_1 x_0 x_0^{-2} x_1 x_0^2 \cdots x_1 x_0^{n-3} x_0^{-(n-2)} x_1 x_0^{n-2} x_1\|. \end{aligned}$$

By the result of Guba's which says the Dehn function of R.Thompson group F is quadratic. We have

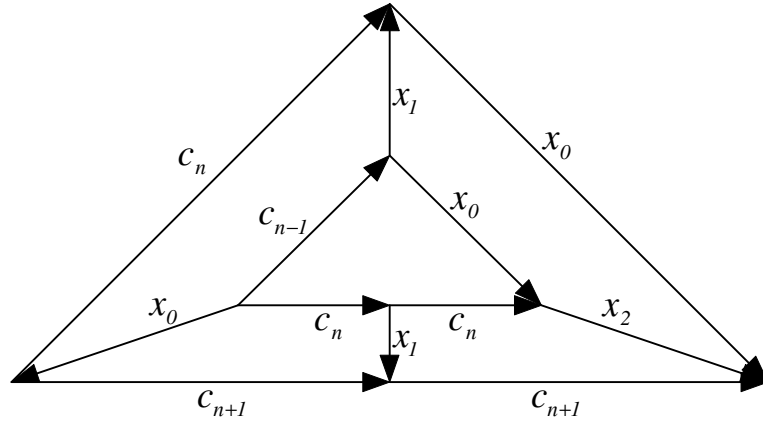
$$\begin{aligned} &\|x_2 x_3 \cdots x_{n-2} x_{n-1} x_1 = x_1 x_3 \cdots x_{n-1} x_n\| \\ &= \|x_0^{-1} x_1 x_0^{-1} x_1 x_0^{-1} \cdots x_1 x_0^{-1} x_1 x_0^{-1} x_1 x_0^{n-2} x_1 = x_1 x_0^{-2} x_1 x_0^{-1} x_1 x_0^{-1} \cdots x_1 x_0^{-1} x_1 x_0^{n-1}\| \\ &= O(n^2). \end{aligned}$$

So we get $h(n, 2) = O(n^2)$. For $k \geq 3$, we consider the following diagram



which shows that $h(n, k) \leq h(n-k+2, 2) + \|x_1^{-(k-1)} x_2 x_1^{k-1} = x_k\| = O(n^2)$ by Guba's result. So we proved (2.3).

For (2.4), let $h(n) = \|c_n x_0 = c_{n+1}^2\|$. Clearly, $h(1) = \|c_1 x_0 = c_2^2\| = 1$. Now let $n \geq 2$, consider the following diagram



shows that $h(n) \leq h(n-1) + h(n, 2)$. So we have that $h(n) = O(n^3)$, which obtains (2.4). \square

Lemma 2.5. Let $n \geq 1$, $1 \leq m \leq n+1$, $0 \leq r, s \leq n$ be integers.

(1)

$$c_n^m x_r = \begin{cases} x_{r-m} c_{n+1}^m, & r \geq m, \\ c_{n+1}^{m+1}, & r = m-1, \\ x_{r+n+2-m} c_{n+1}^{m+1}, & r < m-1; \end{cases} \quad (2.6)$$

(2)

$$x_s^{-1} c_n^m = \begin{cases} c_{n+1}^{m+1} x_{s+m-n-2}^{-1}, & s \geq n+2-m, \\ c_{n+1}^m, & s = n+1-m, \\ c_{n+1}^m, & s \leq n-m; \end{cases} \quad (2.7)$$

(3)

$$c_n^m = x_{n-m+1} c_{n+1}^m; \quad (2.8)$$

(4)

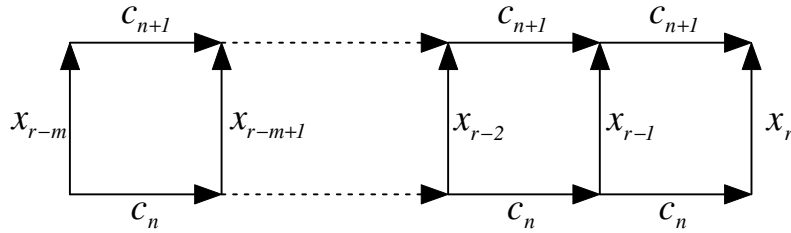
$$c_n^m = c_{n+1}^{m+1} x_{m-1}^{-1}; \quad (2.9)$$

(5)

$$c_n^{n+2} = 1. \quad (2.10)$$

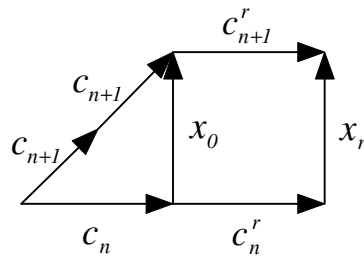
Each of the equalities (2.6) (2.7) (2.8) (2.9) needs $O(n^3)$ elementary steps and (2.10) needs $O(n^4)$ steps.

Proof. For (2.6), in case $r \geq m$ and consider the following diagram

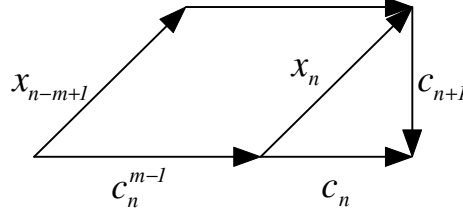


the word is c_n^m labelled by the bottom path, we see that $\| c_n^m x_r = x_{r-m} c_{n+1}^m \| = O(n^3)$.

In case $r = m-1$. If $r = 0$, then $\| c_n x_0 = c_{n+1}^2 \| = O(n^3)$. If $r > 0$, then the following diagram

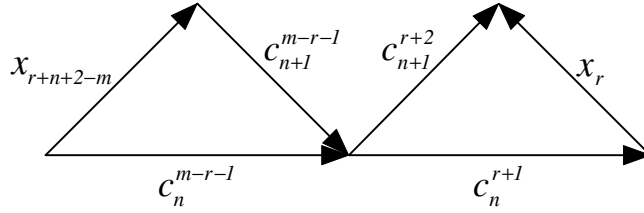


shows that $\|c_n^m x_r = c_{n+1}^{m+1}\| = O(n^3) + O(n^3) = O(n^3)$. For (2.8), Lemma 2.1 and the following diagram



shows that $\|c_n^m = x_{n-m+1} c_{n+1}^m\| = O(n^3)$.

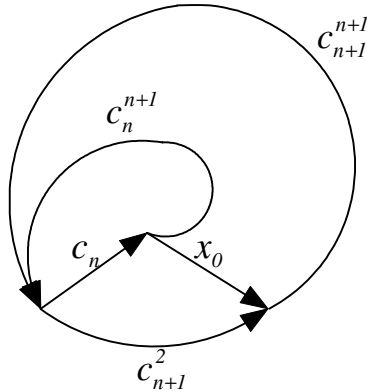
Now we return to prove (2.6), in case $r < m - 1$. We consider the following diagram and apply (2.8) and (2.6)



we have $\|c_n^m x_r = x_{r+n+2-m} c_{n+1}^{m+1}\| = O(n^3)$.

For (2.7) and (2.9), they can be obtained from (2.6) and (2.8). In addition, they all need $O(n^3)$ elementary steps.

Finally, let us prove (2.10). We construct the following diagram



so $\|c_{n+1}^{n+3} = 1\| \leq \|c_n^{n+2} = 1\| + O(n^3)$, which implies $\|c_n^{n+2} = 1\| = O(n^4)$. □

The definition of the complexity ξ which will be occurred in the following lemma can be found in [6] and we improve the corresponding lemma in it.

Lemma 2.11. Let v be a positive word, $n = \xi(v)$. Then there exist a positive word w and an integer $k(1 \leq k \leq n+1)$ such that $c_1 v$ equals $w c_{n+1}^k$ in T and $\|c_1 v = w c_{n+1}^k\| = O(n^4)$, where $\xi(w) \leq \xi(v)$.

Proof. We prove the lemma by induction on $|v|$, which $|v|$ is the length of the word v . It is obviously in the case of $|v| = 0$. Now let $|v| > 0$. Then we can write $v = v' x_j$ for some $j \geq 0$. By the induction assumption, set $n' = \xi(v')$, we have a positive word w' , an integer $k'(1 \leq k' \leq n' + 1)$, such that $\xi(w') \leq n'$ and $\|c_1 v' = w' c_{n'+1}^{k'}\| = O(n^4)$. Consider several cases.

(1) $j > n' + 1$. Applying Lemma 2.5 we have

$$\begin{aligned} c_1 v &= c_1 v' x_j \\ &= w' c_{n'+1}^{k'} x_j \\ &= w' x_{n'+2-k'} c_{n'+2}^{k'} \\ &= \cdots = w' x_{n'+2-k'} \cdots x_{n-k'} c_n^{k'} x_n \\ &= w' x_{n'+2-k'} \cdots x_{n-k'} x_{n-k'} c_{n+1}^{k'}. \end{aligned}$$

We take $w = w' x_{n'+2-k'} \cdots x_{n-k'} x_{n-k'}$. Then we can obtain $\|c_1 v = w c_{n+1}^{k'}\| = O(n^4) + O(n^3) + O(n^4) = O(n^4)$. We now compute the complexity of w

$$\begin{aligned} \xi(w) &= \max\{n - k', \xi(w' x_{n'+2-k'} \cdots x_{n-k'}) + 1\} \\ &= \max\{n - k' + 1, \xi(w' x_{n'+2-k'} \cdots x_{n-k'-1}) + 2\} \\ &= \max\{n - k' + 1, \xi(w' x_{n'+2-k'} \cdots x_{n-k'-2}) + 3\} \\ &= \cdots = \max\{n - k' + 1, \xi(w' x_{n'+2-k'}) + n - n' - 1\} \\ &= \max\{n - k' + 1, \xi(w') + n - n'\} \\ &= n. \end{aligned}$$

(2) $j \leq n' + 1$. In this case, $n = \xi(v) = \xi(v' x_j) = \max\{j, n' + 1\} = n' + 1$.

If $j \geq k'$, we take $w = w' x_{j-k'}$, $k = k'$.

If $j = k' - 1$, we take $w = w'$, $k = k' + 1$.

If $j < k' - 1$, then we take $w = w' x_{j+n'+3-k'}$, $k = k' + 1$.

It is easy to verify by using Lemma 2.5 in all these cases that $\xi(w) \leq n$ and $\|c_1 v = w c_{n+1}^k\| = O(n^4)$. \square

3. PROOF OF THEOREM 1.1

Let w be a word of length n over the alphabet $\{x_0^{\pm 1}, x_1^{\pm 1}, c_1\}$, then we claim that there exists positive words p, q and an integer $m (0 \leq m \leq n+2)$ such that $pc_{n+1}^m q^{-1}$ equals w in T , $\xi(p), \xi(q) \leq n$ and $\|w = pc_{n+1}^m q^{-1}\| = O(n^5)$. If $n = 0$, the claim is trivial. Let $n > 0$. We write $w = yw'$, where $y = x_s^{\pm 1} (s = 0, 1)$ or $y = c_1$. For w' by the inductive assumption, there exist positive words p', q' of complexity $\leq n-1$ and an integer $m' (0 \leq m' \leq n+1)$ such that $w' = p'c_n^{m'}(q')^{-1}$ in T .

(1) $y = x_s (s = 0, 1)$. Let $p = x_s p'$. After a simple calculation we have $\xi(p) = \max\{\xi(p'), |p'| + 1\}$. If $m' = 0$, then $m = 0, q = q'$. If $m' \geq 1$, then by Lemma 2.5 $c_n^{m'} = c_{n+1}^{m'+1} x_{m'-1}^{-1}$. In this case, we set $m = m' + 1, q = q' x_{m'-1}$. $\xi(q) = \max\{m' - 1, \xi(q') + 1\} \leq n$. And we have the following inequality

$$\|w = pc_{n+1}^m q^{-1}\| \leq \|w' = p'c_n^{m'}(q')^{-1}\| + O(n^3).$$

(2) $y = x_s^{-1} (s = 0, 1)$. By [7] and the lemma in [6], there exist a positive word p , integers $t, \delta = 0, 1$ such that $\|x_s^{-1} p' = p x_t^{-\delta}\| = O(n^2)$, $t \leq |p'| + s$, $\xi(p) \leq \xi(p') + 1$. If $m' = 0$, we just take $q = q' x_t^{\delta}$ and $m = 0$. Now let $m' > 0$. If $\delta = 0$, then take $q = q' x_{m'-1}$, $m = m' + 1$. So clearly by Lemma 2.5, $w = x_s^{-1} w' = x_s^{-1} p' c_n^{m'} (q')^{-1} = q c_{n+1}^{m'+1} x_{m'-1}^{-1} (q')^{-1} = p c_{n+1}^m q^{-1}$. If $\delta = 1$.

If $t \geq n + 2 - m'$, then we take $m = m' + 1, q = q' x_{t+m'-n-2}$.

If $t = n + 1 - m'$, then take $m = m', q = q'$.

If $t \leq n - m'$, then we take $m = m', q = q' x_{t+m'}$. It is easy to calculate the complexity of $p, q \leq n$ and we have the following inequality

$$\|w = pc_{n+1}^m q^{-1}\| \leq \|w' = p'c_n^{m'}(q')^{-1}\| + O(n^3).$$

(3) $y = c_1$. In this case, it is similar to be obtained from [6], but we have the inequality more precisely

$$\|w = pc_{n+1}^m q^{-1}\| \leq \|w' = p'c_n^{m'}(q')^{-1}\| + O(n^4).$$

Thus we have proved the claim by induction. Now it is not difficult to get $\|w = 1\| = O(n^5)$, where w is any word of length $\leq n$ on the alphabet $\{x_0^{\pm 1}, x_1^{\pm 1}, c_1^{\pm 1}\}$ and $w = 1$ in T . \square

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